Under the RP-PPS method, the system of prorating is the same but the probabilities are different. Thus,

Terminal branch	Prorated amount
1-2-1-1	$(\frac{4.13}{5.96})(\frac{1.48}{3.80})(3) = .81$
1-2-1-2	$(\frac{4.13}{5.96})(\frac{2.32}{3.80})(3) = 1.27$
1-2-2	$(\frac{1.83}{5.96})(3) = .92$
	Total = 3.00

The estimator $\hat{Y}_i$, Eq. (3.2), can be written in a form that corresponds to the idea of prorating path fruit to terminal branches. Let $p_i = (p_{oi})\ldots(p_{ti})$, which is the probability of selecting the i^{th} terminal branch. It follows that

$$\hat{Y}_i = \frac{y_i}{p_i} \tag{3.3}$$

where $y_i = [(p_{1i})\ldots(p_{ti})y_{oi}]+\ldots+[(p_{(k+i)i})\ldots(p_{ti})y_{ki}]+\ldots+[y_{ti}]$

Thus, y_i is the number of fruit "on" the i^{th} terminal branch including prorated amounts of path fruit. Assuming the RP-PPS method and terminal branch 1-2-1-1 as an example, the value of y_i is $(\frac{4.13}{5.96})(\frac{1.48}{3.80})(3) + 73 = 73.81$ and $\hat{Y}_i$ is $\frac{73.81}{.03103} = 2379$ which gives the same result that was obtained when Eq. (3.2) was used.

Table 3.2, columns headed $\hat{Y}_2$ and $\hat{Y}_4$, present estimates of the total number of apples on the tree for the RP-EP and RP-PPS methods and each of the possible random paths. These estimates were obtained by using the technique of prorating path fruit,

Eq. (3.3). That is, estimates of the total number of apples were obtained by dividing the values of Y_i (last two columns of Table 3.1) by the appropriate probabilities which are presented in Table 3.2, columns P_2 and P_4.

For comparison of the four methods we now need to decide how to include the path fruit for the DS-EP and DS-PPS methods. If the amount of path fruit is small, the best method might be to count all path fruit at the time the tree is mapped to determine terminal branches. In this case, assuming a sample of one terminal branch, the estimator, would be

$$\hat{Y}_i = Y' + \frac{y_i'}{p_i} \qquad (3.4)$$

where Y' is the number of path fruit, y_i' is the number of fruit on the i^{th} terminal branch and p_i is the probability of selecting the i^{th} terminal branch. Alternatives are not considered in this illustration because, from a practical viewpoint, interest is in the random path methods. Thus, as a matter of expediency, the estimator (3.4) was used to obtain the estimates, $\hat{Y}_1$ and $\hat{Y}_3$, that are presented in Table 3.2 for the DS-EP and DS-PPS methods. Since only 51 apples out of 1901 were on path sections, the method of accounting for the apples on path sections probably has a very small impact on the sampling variance.

Exercise 3.3 For terminal branches 3-1-4-1 and 3-3, calculate estimates of the total number of apples on the tree for the DS-EP and DS-PPS methods using the estimator (3.4). Your answer should agree with the estimates that are presented in Table 3.1 for these two branches.

For each terminal branch and each of the four estimators (methods) there is a unique estimate of the total number of apples. All four estimators are unbiased. By definition, an estimator is unbiased if the expected (average) value of the estimates that might occur is equal to the population value. To find the expected value of an estimator, each estimate must be weighted by the probability of its occurence.

Exercise 3.4 For the RP-EP and RP-PPS methods, compute the expected value of the estimates presented in Table 3.2. The answer, except for rounding error, should be exactly 1901, which is the total number of apples on the tree.

3.5 VARIANCES OF THE ESTIMATORS

With reference to the theory of expected values, the variance of a random variable, Y, is the average of the squared deviations of Y from its expected (average) value. To be more specific, suppose Y is a random variable that can equal one of a set of values $Y_1, Y_2, \ldots, Y_N$ with probabilities $P_1, P_2, \ldots, P_N$ where $\Sigma P_i = 1$. By definition, the average value of Y is

$$\bar{Y} = E(Y) = \Sigma P_i Y_i$$

and the variance of Y, which is the average value of $(Y-\bar{Y})^2$, is

$$E(Y-\bar{Y})^2 = \sum_{i}^{N} P_i(Y_i-\bar{Y})^2$$

Exercise 3.5 Show that $\Sigma P_i(Y_i-\bar{Y})^2 = \Sigma P_i Y_i^2 - \bar{Y}^2$

Consider the estimator for the RP-PPS method. It is a random variable that can equal any one of the set of values in column $\hat{Y}_4$ of Table 3.2. The set of probabilities is presented in column P_4. By definition, the variance of the estimator (or estimates) is

$$(.05492)(3751-1901)^2+\ldots+(.06163)(814-1901)^2=800,194$$

or using the right hand side of the equation in exercise 3.5,

$$(.05492)(3751)^2+\ldots+(.06163)(814)^2-(1901)^2 = 800,194$$

The result, 800,194, is the sampling variance for the RP-PPS method when only one terminal branch is selected. If four terminal branches (or random paths) were selected with replacement, four estimates of the tree total would be computed, one for each branch, and the variance of the average of the four estimates would be $\frac{800,194}{4} = 200,048$.

The sampling variances (for a sample of one branch) are presented in Table 3.3 for each of the four methods and each of the six trees. The third tree is the one that was used above as an example. It is not expected that the four methods will always rank in the same order from one tree to another. However, the results illustrate some points that are of interest and importance.

3.6 DISCUSSION OF THE METHODS

The RP-EP method requires considerably less time than the RP-PPS method, but it has relatively high sampling variance because, at any given stage of branching, a large branch has the same probability of selection as a small one. That is, the RP-EP method is such that the probability of selecting a terminal branch has little or no relation to the number of fruit on the branch. The result, as shown by the sampling variances in Table 3.3, is a good illustration of a point that was made earlier. Compared to selecting sampling units with equal probability (as in the DS-EP method), the introduction of unequal probabilities of selection (as in the RP-EP method) will increase the sampling variance unless the selection probabilities are related to the values of the characteristic being measured in a way that will reduce sampling variance.

Figure 3.1 is a dot chart with the number of apples on a branch (column headed EP in Table 3.1) plotted against the values of P_2. The wide range in the selection probabilities and the lack of a relation explains the high sampling variance of the RP-EP method compared with the other methods. For comparison, Figure 3.2 is a dot chart for number of apples and the selection probabilities for the RP-PPS method. Compare Figure 3.2 with Figure 1.2 which showed a dot chart where sampling with pps would rank high.

After a branch has been identified and marked, the time required to obtain its csa, with a convenient instrument that

gives a reading directly in square inches (or square centimeters), is quite small. The use of csa as an auxiliary variable reduced sampling variance by a large amount. The reduction in variance in relation to cost is definitely advantageous. According to Table 3.3, the sampling variances for DS-PPS and RP-PPS are about the same and much less than the sampling variance for the DS-EP method. This indicates that RP-PPS is a good choice because it avoids the work of identifying all terminal branches before sampling. However, results in Table 3.3 should not be accepted as representative. The csa is not always an effective measure. Pruning and maintenance practices, age of trees, species or variety of trees, and other factors have some influence on the relation between csa and number of apples. The purposes of an intensive investigation limited to a few trees include testing different procedures for counting apples or measuring the size of branches, and acquiring ideas that seem to be worth exploring as possibilities for large scale application.

It is extremely important in the processes of sampling to understand the part played by randomization. Important biases sometimes occur even when strict attention is paid to details in making random selections. On the other hand, subjective evaluations or determinations in sampling are commonplace. With knowledge of how various factors effect sampling variance, the exercise of good judgement can be very effective in reducing sampling variance. But, there are points in the processes of

sampling where a determination should be strictly random. Some design constraints may be determined subjectively but selections of units for a sample should be in accord with rigorous, technical interpretation of randomness. It is generally preferable to have random selections made under competent supervision in an office, but that is not always feasible. Thus, one advantage of taking photographs of a sample of bare trees (assuming it is feasible) is that sample branches can be selected in the office. The selected branches are marked on photographs for enumerators. In this situation an enumerator's work is subject to full verification. Incidentally, the economics of sample surveys suggests that larger investments in sample design and selection can often be justified when the same sample is to be used for several surveys rather than one.

Exercise 3.6 Suppose the RP-PPS method is being applied and in the process you come to the following situation:

Assume that branch A, which has a csa equal to 3.2 square inches, has already been selected. It divides into two branches A-1 and A-2 with csa's equal to 1.4 and 1.6. With regard to size, the two branches, A-1 and A-2, qualify as terminal branches and ordinarily A-1 and A-2 would be accepted as terminal branches. But, before selecting one of the two, you happen to notice that A-2 has no apples on it and that A-1 appears to have approximately an average amount. Consider the following alternatives:

(1) *Accept A, which includes A-1 and A-2, as the terminal branch, and expand the count of apples by* $\frac{1}{P_A}$, *where* P_A *was the probability of selecting A.*

(2) *Accept A-1 and A-2 as terminal branches and select one with pps. Expand the count on A-1 or A-2 by* $(\frac{1}{P_A})(\frac{3.0}{1.4})$ *or* $(\frac{1}{P_A})(\frac{3.0}{1.6})$, *depending on whether A-1 or A-2 is selected.*

(3) *Discard A-2 since it has no apples on it and take A-1 as the terminal branch using* $(\frac{1}{P_A})(\frac{3.0}{1.4})$ *as the expansion factor.*

Discuss the alternatives with regard to bias and sampling variance.

Exercise 3.7 Refer to exercise 3.6 and as a variation of the situation assume that branch A-2 has been selected at random in accord with the instructions for the random path method. The enumerator prepares to count the apples on A-2 but finds there are no apples. He recognizes, since a sample of only one branch is to be selected for the sample from this tree, that the estimate of the number of apples on the tree will be zero (assuming no path fruit on the path to A-2). There is obviously a large number of apples on the tree, so he might have a strong opinion that something should be done that would give a better sample. How would you respond to each of the following possibilities:

(1) *Accept A-2 as a terminal branch, which means using zero as an estimate of the number of apples on the tree. Remember A-2 has already been selected.*

(2) *Reject A-2 as a sample. Start at the beginning and select another terminal branch to replace A-2.*

(3) *Accept A which includes A-1 and A-2, as the terminal branch for the sample.*

Discuss the three possibilities with regard to bias and sampling variance.

Exercise 3.8 In application of the RP-PPS method would it be advisable to be looking forward, as one approaches the terminal branch stage, for branches that are large enough to be terminal branches but clearly have a very small number of apples on them. With reference to the diagram in exercise 3.6 as an example, an enumerator looking forward, and considering what was ahead, could have stopped when A was selected and accepted A as a terminal branch. Otherwise, he would normally have followed the selection procedure one stage further. In application of the random path method, what is your opinion of the feasibility of looking ahead and taking eye estimates of numbers of apples into account in determining the terminal branch. Can it be used to reduce sampling error without risk of introducing bias? Think about the matter with regard to instructions that would be given to enumerators.

Exercise 3.9 It is not likely that there would be an interest in estimating the average number of terminal branches per tree. However, as an exercise, suppose the RP-PPS method is applied to the tree for which data are presented in Table 3.1. Assume that the following four terminal branches are selected as a sample:

1-1-2, 1-2-1-2, 2-4, and 3-2-1. From this sample, estimate the number of terminal branches on the tree. (The selection probabilities have already been computed, see Table 3.2). The parameter being estimated is 26. Ans. 33.4.

Exercise 3.10 Suppose a sample of 25 apple trees has been selected and that four enumerators have been trained in the application of the RP-PPS method. Assume that each enumerator, working independently and using the RP-PPS method, selects a sample of one terminal branch from each of the 25 trees. It is unlikely that enumerators will interpret terminal branches in exactly the same way. For example, one enumerator might have a tendency to follow the random path to terminal branches of the smallest permissible size, whereas another might stop as soon as he obtains a branch that is small enough to qualify as a terminal branch. Or, a branch along a path might be treated as a terminal branch by one enumerator and as path fruit by another. However, for each enumerator an estimate of the total number of apples on each tree is made using either (3.2) or (3.3) as the estimator. The 25 estimates are added together to obtain an estimate of the total number of apples on the 25 trees. This gives four estimates, one for each enumerator, of the total number of apples on the 25 trees.

(a) Assume that random selection is performed correctly at each stage of branching (after all branches at the stage have been completely identified and measured), and assume that apples have

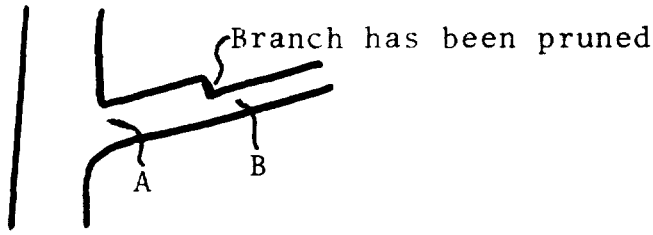
been correctly counted. Do the four estimates of the total number of apples all have the same expected value and the same variance?

(b) Suppose four estimates, one for each enumerator, of the total number of terminal branches on the 25 trees are made. Do these estimates have the same expected value? Why?

(c) Two enumerators measuring the csa's of any given set of branches are not likely to obtain exactly the same numerical values. Is this important? Discuss.

(d) The assumptions made in (a) are subject to question. Try listing some differences among enumerators that will, and will not, have an effect on the expected value of an estimate of the total number of apples on the 25 trees.

Exercise 3.11 *Suppose, owing to pruning practices, that many cases like the following are found:*



Assume the instructions were to always measure the csa at the base (point A) of a branch, Would you expect the csa measurements under the RP-PPS method to be ineffective, or even increase the sampling variance, compared with the DS-EP method? In cases like the above drawing, perhaps measuring the csa at position B would be more effective. What is your opinion? Incidentally, this is

a good example of why it is essential that a research and development staff should have actual experience with practical operations and decisions that must be made by enumerators. Do not expect high quality results when instructions are not well adapted. Agreement between concepts (the theoretical model) and operations as actually performed is of fundamental importance.

Table 3.1--Data by Branches for Apple Tree No. 3

Branch identification	1st stage		2nd stage		3rd stage		4th stage		No. of apples on or assigned to a terminal branch	
	csa	No.	csa	No.	csa	No.	csa	No.	EP	PPS
1-1-1	11.60		3.65		2.68	206			206	206
1-1-2					.97	32			32	32
					3.65					
1-2-1-1			5.61	(3)	4.13		1.48	73	73.8	73.8
1-2-1-2							2.32	138	138.7	139.3
							3.80			
1-2-2					1.83	133			134.5	133.9
					5.96					
1-3-1			2.01		.97	32			32	32
1-3-2					1.03	30			30	30
					2.00					
1-4			1.43	27					27	27
1-5			2.24	88					88	88
			14.94							
2-1-1	13.45	(6)	3.36		.92	42			42.8	42.5
2-1-2					1.99	109			109.7	110.1
					2.91					
2-2-1			5.09		1.47	74			74.7	74.7
2-2-2-1					3.47	(16)	1.64	56	64.4	65.2
2-2-2-2							1.54	116	124.4	124.6
					4.94		3.18			
2-3			1.99	124					125.5	125.0
2-4			1.83	79					80.5	79.9
			12.27							
3-1-1	12.84	(1)	6.30	(2)	1.47	30			30.6	30.4
3-1-2					1.21	31			31.6	31.3
3-1-3					1.91	41			41.6	41.5
3-1-4-1					4.13	(23)	1.47	16	27.7	29.6
3-1-4-2							1.15	23	34.8	33.6
					8.72		2.62			
3-2-1			5.35		1.40	35			35.1	35.1
3-2-2					1.42	61			61.1	61.1
3-2-3					1.76	116			116.1	116.1
3-2-4					3.26	88			88.1	88.1
					7.84					
3-3			2.59	50					50.3	50.2
	37.89		14.24						1901.0	1901.0

Total number of apples on terminal branches 1850
Total number of apples on path sections 51
Grand total 1901

Table 3.2 Probabilities of Selection and Estimates of the Total Number of Apples on Tree No. 3

Terminal branch no.	DS-EP		RP-EP		DS-PPS		RP-PPS	
	P_1	$\hat{Y}_1$	P_2	$\hat{Y}_2$	P_3	$\hat{Y}_3$	P_4	$\hat{Y}_4$
1-1-1	.03846	5407	.03333	6180	.06095	3431	.05492	3751
1-1-2	.03846	883	.03333	960	.02206	1502	.01988	1610
1-2-1-1	.03846	1949	.01667	4425	.03366	2220	.03103	2379
1-2-1-2	.03846	3639	.01667	8325	.05276	2667	.04864	2863
1-2-2	.03846	3509	.03333	4035	.04162	3247	.03530	3794
1-3-1	.03846	883	.03333	960	.02206	1502	.01998	1602
1-3-2	.03846	831	.03333	900	.02342	1332	.02121	1414
1-4	.03846	753	.06667	405	.03252	881	.02930	921
1-5	.03846	2339	.06667	1320	.05094	1776	.04590	1917
2-1-1	.03846	1143	.04167	1026	.02092	2059	.03073	1384
2-1-2	.03846	2885	.04167	2634	.04526	2459	.06647	1657
2-2-1	.03846	1975	.04167	1794	.03343	2265	.04382	1706
2-2-2-1	.03846	1507	.02083	3090	.03730	1552	.05334	1222
2-2-2-2	.03846	3067	.02083	5972	.03502	3363	.05009	2489
2-3	.03846	3275	,08333	1506	.04526	2791	.05757	2171
2-4	.03846	2105	.08333	966	.04162	1949	.05294	1509
3-1-1	.03846	831	.02778	1101	.03343	948	.02528	1203
3-1-2	.03846	857	.02778	1137	.02752	1147	.02081	1506
3-1-3	.03846	1117	.02778	1497	.04344	995	.03284	1264
3-1-4-1	.03846	467	.01389	2001	.03343	530	.03984	742
3-1-4-2	.03846	649	.01389	2505	.02615	931	.03117	1078
3-2-1	.03846	961	.02778	1263	.03184	1150	.02274	1542
3-2-2	.03846	1637	.02778	2199	.03229	1940	.02306	2648
3-2-3	.03846	3067	.02778	4179	.04003	2949	.02858	4062
3-2-4	.03846	2339	.02778	3171	.07414	1238	.05294	1665
3-3	.03846	1351	.11111	453	.05890	900	.06163	814
	.99996		1.00001		.99997		1.00001	

Table 3.3 Variances of Estimates of the Total Number of Apples on Each of Six Trees from a Sample of One Terminal Branch

Tree	No. of terminal branches	csa of trunk	No. of apples on tree	Variances			
				DS-EP	RP-EP	DS-PPS	RP-PPS
				(000)	(000)	(000)	(000)
1	13	7.0	214	40	28	24	22
2	27	20.0	1448	882	1383	674	478
3	26	23.0	1901	1419	2815	755	800
4	20	16.5	1658	1148	1444	380	350
5	19	13.5	403	82	263	65	79
6	30	19.5	1575	894	4339	416	513
Total	135	99.5	7199	4465	10272	2314	2242

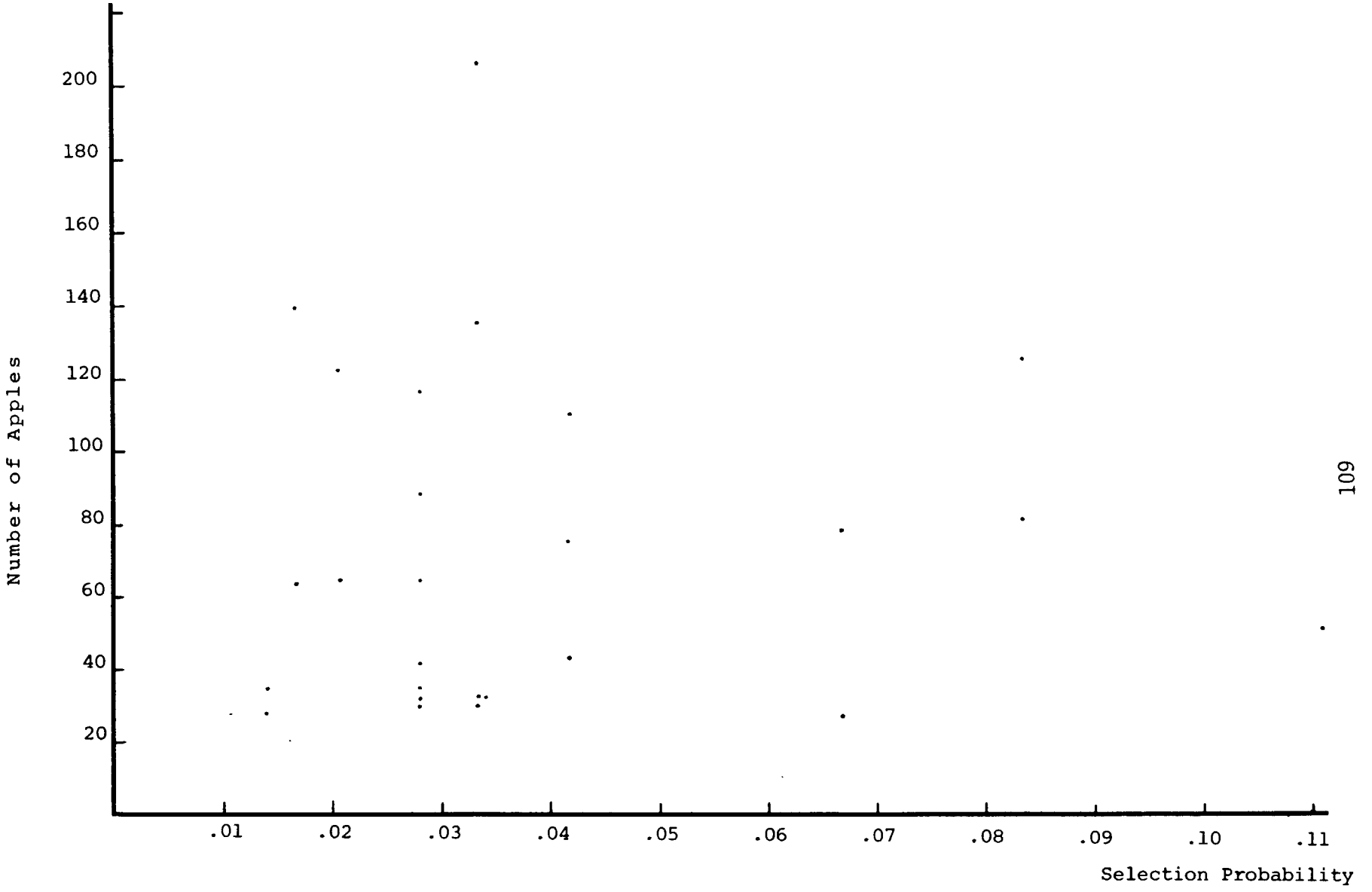


Figure 3.1 Dot Chart---Number of Apples vs Selection Probabilities for RP-EP

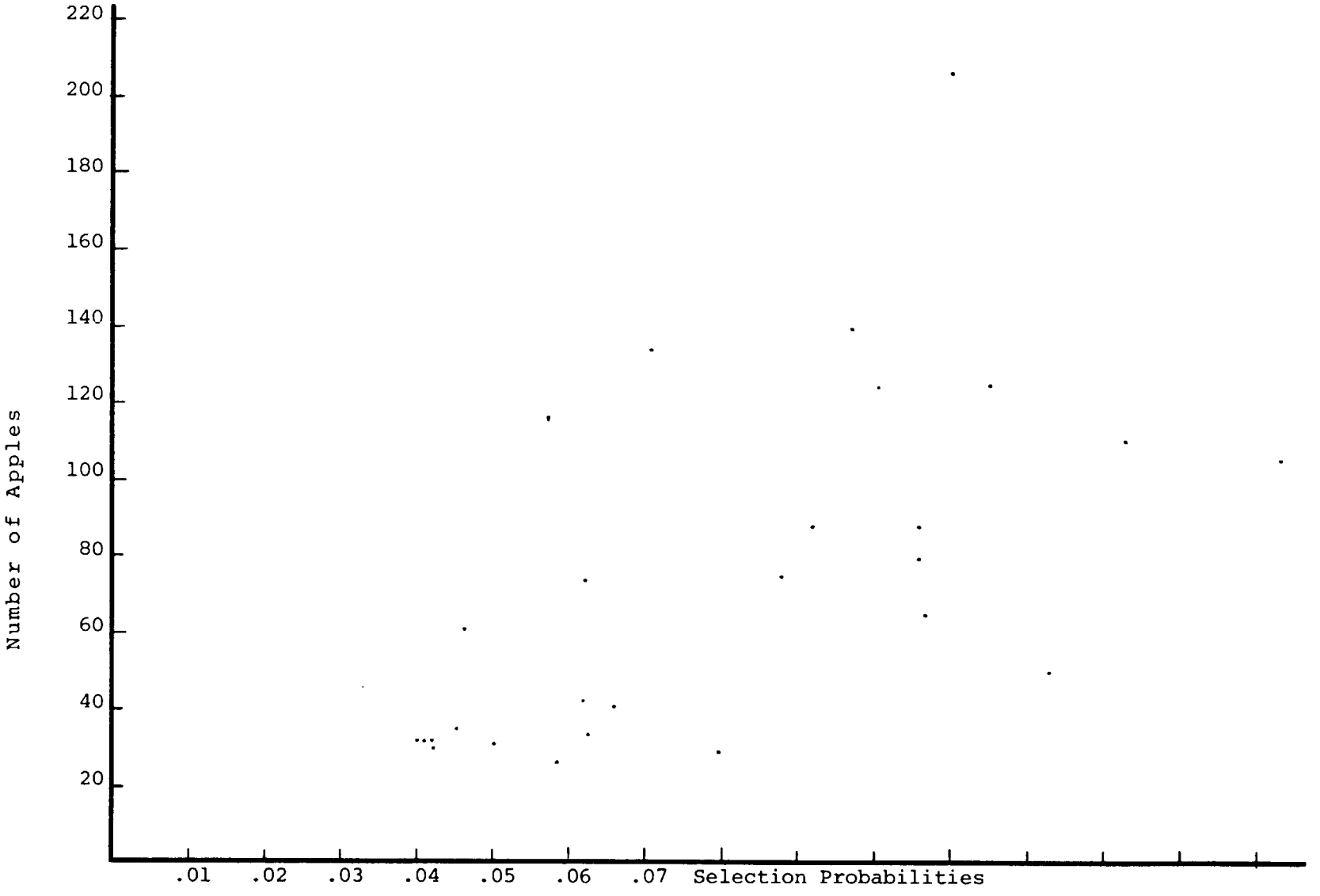


Figure 3.2 Dot Chart---Number of Apples vs Selection Probabilities for RP-PPS

TWO-STAGE SAMPLING

CHAPTER IV

4.1 INTRODUCTION

Most sampling plans for estimating or forecasting tree-crop production will involve three or four stages of sampling. Typically, there will be a sample of orchards (fields), a sample of trees in selected orchards, and a sample of branches from a sample of trees. Fruit on the sample branches would be counted and a small sample of fruit on the sample branches might be selected for measurements of size of fruit.

This chapter illustrates some alternative two-stage sampling plans using data for the six apple trees. Trees are the psu's (primary sampling units) and terminal branches or "paths" are the ssu's (secondary sampling units). The six trees will be treated as a population to be sampled and population variance formulas will be used to find the first and second-stage components of variance. Incidentally, the problem of making accurate counts of numbers of fruit on sample branches needs serious consideration. However, in the illustrations that follow, attention is limited to matters of sampling.

In the application of two-stage sampling, psu's are often selected with probabilities proportional to N_i, where N_i is the number of ssu's in the i^{th} psu. For some surveys, sampling with

probability proportional to N_i has important advantages. When the N_i are not known, approximations of N_i are often used.

With regard to sampling trees, the N_i (number of branches on trees) are not known and it is not feasible to determine the N_i for trees in an orchard. Some other effective measure of size must be found or the sample trees will need to be selected with equal probability. One possibility is to use a double sampling procedure. For example, a "large" sample of trees might be selected with equal probabilities. For each tree in the large sample a measurement of size, that takes relatively little time, might be made and used in the selection of a small sample of trees from the large sample. Possible measures of size are the csa of the trunk, the sum of the csa's of primary branches, and eye estimates of the amount of fruit. The feasibility of double sampling would depend upon the cost of obtaining the measurements of size and the relation between the measure of size and the amount of fruit on the trees. Stratification of trees within an orchard also needs to be considered. Sometimes strata within an orchard are readily recognized; for example, differences in age or variety. Perhaps a relation between size of trunk and number of apples will be found to be effective only within strata comprised of trees of the same variety and of a uniform condition.

Stratification, systematic sampling, or other techniques might be applied at any stage of sampling. However, for simplicity, the discussion will be limited to: (1) simple random sampling of psu's (selection with equal probability and without replacement)

and (2) sampling the psu's with pps (sampling with unequal probabilities of selection and replacement). Within each selected psu we will assume that a simple random sample of n_i ssu's is selected. The number of psu's in the sample is m and the number of ssu's in the sample is $n = \sum_{i}^{m} n_i$.

Refer to Table 4.1 for an exposition of the notation that will be used for representing data for a population. Examine the notation carefully. Sample data are represented in the same way except that lower case letters are used.

Since a general mathematical formulation of estimators and their variances is rather complex for two-stage sampling, we will procede from specific cases to more general description. The primary purpose of the next section is to present an elementary view of two-stage sampling.

4.2 PRIMARY SAMPLING UNITS EQUAL IN SIZE

The simplest case of two-stage sampling is one where all psu's have the same number of ssu's, where simple random is applied at both stages, and where the same number of ssu's is selected from each psu in the sample. In this case, and with reference to the notation in Table 4.1, the N_i all equal $\bar{N}$ and the n_i all equal $\bar{n}$. To summarize, the sampling plan under consideration is to select a simple random sample of m psu's from a population of M psu's and a simple random sample of $\bar{n}$ ssu's from each of the m psu's, which gives a total sample of $n = m\bar{n}$ ssu's.

For illustration a hypothetical population of 4 psu's with 5 ssu's in each is assumed. The 20 values of Y_{ij} are presented in the top part of Table 4.2. Deviations of Y_{ij} from $\bar{\bar{Y}}$ are also presented. In single-stage sampling, there is one component of variance, namely the variance of $(Y_{ij} - \bar{\bar{Y}})$ which in the illustration is 487.053.

In two-stage sampling, each deviation $(Y_{ij} - \bar{\bar{Y}})$ divides into two deviations as follows:

$$(Y_{ij} - \bar{\bar{Y}}) = (\bar{\bar{Y}}_i - \bar{\bar{Y}}) + (Y_{ij} - \bar{\bar{Y}}_i)$$

The values of $(\bar{\bar{Y}}_i - \bar{\bar{Y}})$ are a set of deviations which reflect the variation among psu's and the values of $(Y_{ij} - \bar{\bar{Y}}_i)$ form the other set which reflects variation among ssu's within psu's. Turn to Table 4.2 and verify the deviations (components) $(\bar{\bar{Y}}_i - \bar{\bar{Y}})$ and $(Y_{ij} - \bar{\bar{Y}}_i)$. Notice that the between psu component, $(\bar{\bar{Y}}_i - \bar{\bar{Y}})$, varies from one psu to another but is constant within a psu. There are only M different values of $(\bar{\bar{Y}}_i - \bar{\bar{Y}})$ and selecting a sample of m psu's is equivalent to selecting a sample of m values of $(\bar{\bar{Y}}_i - \bar{\bar{Y}})$. Also, study the values of the within psu component, $(Y_{ij} - \bar{\bar{Y}}_i)$. It varies from one ssu to another within a psu, but its average value is zero for each psu. Therefore, these deviations reflect only variation within psu's. The second stage of sampling is equivalent to selecting $m\bar{n}$ of the deviations, $(Y_{ij} - \bar{\bar{Y}}_i)$

Now consider the variance of $\bar{\bar{y}}$, the mean of a two-stage sample. The difference between $\bar{\bar{y}}$ and $\bar{\bar{Y}}$ may be expressed as follows:

$$\bar{\bar{y}} - \bar{\bar{Y}} = \bar{\bar{d}}_1 + \bar{\bar{d}}_2$$

where $\bar{\bar{d}}_1$ is the average value of $(\bar{\bar{Y}}_i - \bar{\bar{Y}})$ for the m psu's in the sample and d_2 is the average value of $(Y_{ij} - \bar{\bar{Y}}_i)$ for the $m\bar{n}$ ssu's in the sample.

Exercise 4.1 With reference to Table 4.2, suppose that psu's 1 and 3 are selected at the first stage and that ssu's 1 and 4 are selected within psu No. 1 and ssu's 3 and 5 are selected within psu No. 3. Find the values of $\bar{\bar{y}}$, $\bar{\bar{d}}_1$, *and* $\bar{\bar{d}}_2$. *Verify that* $\bar{\bar{y}} - \bar{\bar{Y}} = \bar{\bar{d}}_1 + \bar{\bar{d}}_2$. *Ans.* 34-43 = -9.4 + 0.4.

Since $\bar{\bar{d}}_1$ is the average of m random values of $(\bar{\bar{Y}}_i - \bar{\bar{Y}})$ and $\bar{\bar{d}}_2$ is the average of $m\bar{n}$ random values of $(Y_{ij} - \bar{\bar{Y}}_i)$, it follows that $\bar{\bar{d}}_1$ and $\bar{\bar{d}}_2$ are random variables. It happens that $\bar{\bar{d}}_1$ and $\bar{\bar{d}}_2$ are independent. Therefore, the variance of $\bar{\bar{y}}$ is equal to the variance of $\bar{\bar{d}}_1$ plus the variance of $\bar{\bar{d}}_2$. From knowledge of the variance of the mean of a simple random sample, one might anticipate what the variances of $\bar{\bar{d}}_1$ and $\bar{\bar{d}}_2$ are and hence the formula for the variance of $\bar{\bar{y}}$ which is:

$$V(\bar{\bar{y}}) = \frac{M - m}{M} \frac{S_1^2}{m} + \frac{\bar{N} - \bar{n}}{\bar{N}} \frac{S_2^2}{m\bar{n}} \qquad (4.1)$$

where S_1^2 is the variance of $(\bar{\bar{Y}}_i - \bar{\bar{Y}})$ and S_2^2 is the variance of the deviations $(Y_{ij} - \bar{\bar{Y}}_i)$. In this case, S_2^2 is a simple average of the within psu variances, S_{2i}^2, which is logical since the psu's are equal in size and are selected with equal probabilities. Moreover, the within psu sample size is constant.

For the illustration, values of S_1^2 and S_2^2 as functions of the deviations $(\bar{\bar{Y}}_i - \bar{\bar{Y}})$ and $(Y_{ij} - \bar{\bar{Y}}_i)$ are shown at the bottom of Table 4.2.

In practice, the two sets of deviations $(\bar{\bar{Y}}_i - \bar{\bar{Y}})$ and $(Y_{ij} - \bar{\bar{Y}}_i)$, would not be computed. The variances, S_1^2 and S_2^2, could be calculated as follows:

$$S_1^2 = \frac{1}{\bar{N}^2} \left[\frac{\Sigma Y_i^2 - \frac{Y^2}{M}}{M - 1} \right] \tag{4.2}$$

$$S_2^2 = \frac{1}{M(\bar{N}-1)} \left[\Sigma\Sigma_{ij} Y_{ij}^2 - \frac{\Sigma Y_i^2}{\bar{N}} \right] \tag{4.3}$$

Exercise 4.2 Use Eq.'s 4.2 and 4.3 to find the values of S_1^2 *and* S_2^2 *in the numerical example. Explain why* $\bar{N}^2$ *appears as a divisor in Eq. 4.2.*

Exercise 4.3 For $m=2$ *and* $\bar{n}=2$ *find the variance of* $\bar{\bar{y}}$ *using Eq. 4.1. Ans.*118.9

Exercise 4.4 Show algebraically, that the right hand side of Eq. 4.3 is equal to $\frac{\Sigma_i S_{2i}^2}{M}$, *where* S_{2i}^2 *is the variance among ssu's within the* i^{th} *psu.*

One partial check on a variance formula is to determine whether it reduces to known formulas for special cases. Two special cases are of interest: (1) When m = M, two-stage sampling becomes stratified random sampling. That is, the psu's become strata. Observe, when m = M, that the first term on the right side of Eq. 4.1 vanishes and the second term becomes the variance for a stratified random sample of $\bar{n}$ units from each stratum (psu). (2) When $\bar{n} = \bar{N}$, two-stage sampling reduces to single-stage cluster sampling. In this case the last term in Eq. 4.1 vanishes, leaving the first

term which is the variance for a cluster sample where the clusters (sampling units) are the psu's.

Exercise 4.5 Suppose m=1 *and* $\bar{n}$=1. *In this case the selection of one psu at random and the selection of one ssu within it is equivalent to a single-stage sample of one ssu. Therefore, the variance of* $\bar{\bar{y}}$ *given by Eq. 4.1 when* m=1 *and* $\bar{n}$=1 *should be equal to the variance of* $\bar{\bar{y}}$ *for a single-stage random sample when* n=1. *Verify this using the data in Table 4.2. Remember the appropriate variance formula for the single-stage sample is Eq. 1.4.*

It is important to study the structure of the variance formula, Eq. 4.1, for the variance of $\bar{\bar{y}}$. When the number of psu's in the sample is fixed, increasing the size of the sample in each psu reduces only the second component of variance. As $\bar{n}$ increases, a point is reached where the among-psu variance is the major component and further increases in $\bar{n}$ contributes very little to reducing the variance of $\bar{\bar{y}}$. Notice that increasing m reduces both components when n_i is constant for all psu's.

4.3 PRIMARY SAMPLING UNITS UNEQUAL IN SIZE

Populations having psu's with equal numbers of ssu's are relatively infrequent. In this section, it is assumed that the numbers, N_i, of ssu's vary and that simple random sampling (without replacement) is applied at both stages.

As discussed in Chapter I, Sec. 1.1.2, "P" or "p" with appropriate subscripts refer to selection probabilities on the occasion of a particular random draw and "f" with an appropriate subscript refers to the probability that a particular unit has of being in the sample.

A general expression for the probability, f_{ij}, which any given ssu has of being included in a two-stage sample is:

$$f_{ij} = f_i f(j|i) \tag{4.4}$$

where f_i is the probability which the i^{th} psu has of being in the sample, and

$f(i|j)$ is the conditional probability which the j^{th} ssu in the i^{th} psu has of being in the sample, given that the i^{th} psu is in the sample of psu's.

With simple random sampling at both stages, $f_i = \frac{m}{M}$, and $f(j|i) = \frac{n_i}{N_i}$. Since f_i is constant for the case under consideration, let $f_i = f_1$ which is the sampling fraction at the first stage. Also, let $f(j|i) = f_{2i}$ which is the sampling fraction at the second stage within the i^{th} psu. Then Eq. 4.4 reduces to:

$$f_{ij} = f_1 f_{2i} \tag{4.5}$$

If the f_{2i} (the sampling fractions at the second stage) are constant, f_{ij} is constant and every ssu in the population has the same chance of being in the sample. Then, Eq. 4.5 becomes:

$$f = f_1 f_2$$

where f_2 is the constant second-stage sampling fraction. However, in the interest of generality, a requirement that f_{2i} be constant will not be specified at this point in the discussion.

An estimator of the population mean, $\bar{\bar{Y}}$, is

$$\hat{\bar{y}} = \left(\frac{1}{N}\right)(M) \sum_{i}^{m} \frac{N_i \bar{\bar{y}}_i}{m} \tag{4.6}$$

where $\bar{\bar{y}}_i = \frac{\sum_j^{n_i} y_{ij}}{n_i}$ is the average of n_i ssu's in the sample from the i^{th} psu in the sample. Study the estimator 4.7 and observe that:

$N_i\bar{\bar{y}}_i$ is an estimate of Y_i, the total for the i^{th} psu;

$\sum_i^m \frac{N_i\bar{\bar{y}}_i}{m}$ is an average of the estimated totals for the m psu's in the sample; therefore,

$(M)\sum_i^m \frac{N_i\bar{\bar{y}}_i}{m}$ is an estimate of the population total and $(\frac{1}{\bar{N}})$ in Eq. 5.6, changes the estimated total to an estimate of $\bar{\bar{Y}}$.

The variance of $\hat{y}$ is given by:

$$V(\hat{y}) = \frac{1}{m}\left[(1-f_1)S_1^2 + \frac{1}{N^2}\sum_i^M M(1-f_{2i})\frac{N_i^2S_{2i}^2}{n_i}\right] \qquad (4.7)$$

where $S_1^2 = \frac{1}{\bar{N}^2}\frac{\sum_i^M(Y_i - \bar{Y})^2}{M-1}$ is the variance among psu totals divided by $\bar{N}^2$ so S_1^2 will be expressed on the basis of one ssu, and

$S_{2i}^2 = \frac{\sum_j^{N_i}(Y_{ij} - \bar{\bar{Y}}_i)^2}{N_i - 1}$ is the variance among ssu's within the i^{th} psu.

The first part of 4.7, $\frac{1}{m}(1-f_1)S_1^2$, is the variance of $\hat{y}$ assuming all of the m psu's are enumerated completely. That is, the theory for single-stage sampling applies to the first stage.

The quantity:

$$(1 - f_{2i}) \frac{N_i^2 S_{2i}^2}{n_i}$$

in Eq. 4.7 is recognizable as the variance of $N_i \bar{\bar{y}}_i$ where $\bar{\bar{y}}_i$ is the mean of a simple random sample of n_i ssu's in the i^{th} psu.

Eq. 4.7 was written in the above form for comparison with other variance formulas given later for two-stage sampling. The second term within [ ] could be written as follows:

$$\left(\frac{1}{\bar{N}^2}\right)\left(\frac{1}{M}\right)\sum_i^M (1 - f_{2i}) \frac{N_i^2 S_{2i}^2}{n_i} \qquad (4.8)$$

because $\frac{M}{N^2} = \frac{1}{\bar{N}^2} \frac{1}{M}$. Expression 4.8 shows that the variances of $N_i \bar{\bar{y}}_i$ are summed over all psu's in the population and the sum is divided by M giving an average of such variances. The variances of $N_i \bar{\bar{y}}_i$ receive equal weight in the average because the psu's are selected with equal probabilities. Since the average variance of $N_i \bar{\bar{y}}_i$ pertains to psu totals, the divisor $\bar{N}^2$ appears in 4.8 to convert the variance to a basis of one ssu. Such analysis of a formula is helpful in determining whether one has the right formula for a particular purpose.

Exercise 4.6 If the variance formulas (4.1) and (4.7) are correct, formula (4.7) should reduce to (4.1) when $N_i = \bar{N}$ *and* $n_i = \bar{n}$. *Show that this is true.*

When the second-stage sampling fractions $\frac{n_i}{N_i}$, are constant and equal to f_2, the estimator, (4.6), reduces to:

$$\hat{y} = \frac{\Sigma\Sigma y_{ij}}{f_2 m\bar{N}} \tag{4.9}$$

and its variance, (4.7), reduces to:

$$V(\hat{y}) = (1 - f_1)\frac{S_1^2}{m} + (1 - f_2)\frac{S_2^2}{m\bar{n}} \tag{4.10}$$

where S_1^2 is the same as in 4.7,

$$S_2^2 = \sum_i^M \frac{N_i}{N} S_{2i}^2,$$

and $$\bar{n} = \frac{\sum_i^M n_i}{M} = \frac{\Sigma f_2 N_i}{M} = f_2\bar{N}$$

Exercise 4.7. Show that Eq.'s (4.9) and (4.10) follow from (4.6) and (4.7) when $f_2 = \frac{n_i}{N_i}$

Exercise 4.8 Show that $f_2 m\bar{N}$, *in Eq. 4.9, is equal to the expected sample size. That is, show that* $E(n) = f_2 m\bar{N}$ *where* $n = \sum_i^m n_i$. *In practice one would probably use* n, *the actual sample size, in the estimator instead of the expected size,* $f_2 m\bar{N}$. *Moreover,* $\bar{N}$ *is not known in most practical applications.*

4.3.1 NUMERICAL EXAMPLE

As a numerical example, the apple tree data presented in Table 2.1 will be treated as a population to be sampled. The psu's are trees and ssu's are terminal branches. The number of trees in an orchard is usually large and in practice the number of sample trees selected from an orchard would be relatively small, that is $(1 - f_1)$ would be nearly equal to 1. Accordingly, for this illustration, $(1 - f_1)$ is assumed to be 1 even though $M = 6$ and $(1 - f_1) = \frac{M-m}{M}$ is considerably less than 1.

Suppose we are interested in knowing what the sampling variance is for the following three allocations of a sample of four terminal branches assuming simple random sampling at both stages:

Allocation	No. of Trees m	No. of Branches Selected from Each Tree $n_i = \bar{n}$
1	1	4
2	2	2
3	4	1

To find the variances for the three allocations we need part of the results in Table 2.6. The relevant results, N_i, Y_i, and S_{2i}^2, from Table 2.6 are included in Table 4.3 along with some other information that will be used later.

In each allocation, n_i is constant (the same for all trees) which means that $\frac{n_i}{N_i}$ is not constant and the branches do not have equal probability of being in the sample. Thus, the estimator, Eq. 4.6, and its variance, Eq. 4.7, are applicable. The variances for the three allocations are presented in Table 4.4.

Exercise 4.9 Refer to the data presented in Table 4.3, columns N_i, Y_i, *and* S_{2i}^2 *and perform the calculations that are needed to obtain the results presented in Table 4.4 for* $m = 2$ *and* $n_i = \bar{n} = 2$. *Assume that* f_1 *is negligable.*

Exercise 4.10 Complete the following table:

				Variance Components	
n	*m*	$\bar{n}$	$V(\hat{y})$	*Among psu's*	*Within psu's*
1	1	1	1167.0		
2	1	2			
4	1	4			
2	2	1			
4	2	2			
8	2	4			
4	4	1			
8	4	2			
16	4	4	306.5		

If you understand the variance formula 4.7 and the results in Table 4.3, this table can be completed very easily. First, fill in the "Among psu's" column by copying the appropriate numbers from Table 4.4. Consider how to fill in the "Within psu's" column by making simple changes in the within psu components in Table 4.4. Study the results. For a constant value of $\bar{n}$ *and an increase in m from 1 to 4 there is a 75 percent reduction in the variance of* $\hat{y}$; *but, for a constant m, increasing* $\bar{n}$ *from 1 to 4 reduces the variance of* $\hat{y}$ *by less than 50 percent.*

Exercise 4.11 One of the numbers in Table 4.4 is the sampling variance for m = 2 and $n_i = N_i$. What is the number?

Exercise 4.12 Find the probability that any given terminal branch on tree No. 1 has of being in the sample when m = 2 and $n_i = 2$ for all trees. What is the probability for tree No. 3? Is the unequal probability something to be concerned about? In what ways?

It is of interest to compare the variance for a simple random (single-stage) sample of 4 branches with the variances of $\hat{y}$ in Table 4.4. The variance among the 135 branches is 1,762 (see Table 2.6). Hence, the variance of the mean of a sample of 4 branches is $\frac{1762}{4} = 440$, disregarding the fpc. The answer, 440, is less than the variances of $\hat{y}$ in Table 4.4. This is expected with the possible exception of the allocation m = 4 and $n_i = 1$, which has a variance equal to 583.5. However, when one recognizes in the specified two-stage plans that all branches do not have the same probabilities of selection, it is resonable to expect that the answer for simple random sampling would be less than 583.5.

Suppose we wish to give every branch an equal chance of being in the sample. Considering samples of 4 branches the overall sampling fraction would be $\frac{4}{135}$. If we specify that m = 2, then $f_1 = \frac{1}{3}$ and all $\frac{n_i}{N_i}$ (or f_2) should equal $\frac{4}{45}$. Since the N_i are small and the n_i must be integers, it is not possible to have all $\frac{n_i}{N_i}$ exactly equal to $\frac{4}{45}$. This presents a type of practical problem that often occurs when working with small integers. Ways

of dealing with this problem will not be discussed at this point. Instead, we will procede as though the fraction $\frac{n_i}{N_i}$ is sufficiently close to $\frac{4}{45}$ to warrant use of the unweighted average of the sample data as the estimator and the variance formula 4.10. Assuming $(1 - f_1) = 1$, for reasons explained above, and substituting the numerical values of S_1^2 and S_2^2 in 4.10, we have:

$$V(\hat{y}) = \frac{1}{m}\left[917.1 + (1 - f_2)\frac{1367}{\bar{n}}\right] \qquad (4.11)$$

When $m = 2$ and $f_2 = \frac{4}{45}$, the value of $\bar{n}$ is 2 and the variance of $\hat{y}$ is 769.9. This answer compares with 797.6 in Table 4.4.

Exercise 4.13 Verify the numbers, 917.1 and 1367, in Eq. 4.11.

It is often desirable to specify that all ssu's in the population have an equal chance of being in the sample. As discussed above, one way of fulfilling this requirement is to select psu's with equal probability and apply a constant sampling fraction at the second stage of sampling. But, when the sizes of the pus's vary widely, this method often has two important disadvantages: (1) Variance associated with variation in the sizes of the psu's is included in the variance of an estimate unless such variation is reduced by design. Notice that S_1^2 in 4.7 is the variance among psu totals rather than the variance among psu means. Incidentally, an auxiliary variable(s) might be useful in reducing the sampling variance associated with the first stage of sampling. (2) When the second-stage sampling

fraction, $\frac{n_i}{N_i}$, is constant, n_i is proportional to N_i and the workload varies from one psu to another. For many surveys, it is important for reasons of economy that n_i, rather than $\frac{n_i}{N_i}$ be constant. Selecting psu's with pps is often very helpful in overcoming these disadvantages.

Exercise 4.14 Under the plan of applying a sampling fraction of $\frac{4}{45}$ *to each tree that is selected, suppose that trees numbered 1 and 3 are selected. Find the values of* n_i *for these two trees where* n_i *is* $\frac{4}{45} N_i$ *rounded to the nearest integer. Also, find* $n = \Sigma n_i$. *Do the same assuming trees numbered 2 and 4 are selected. This illustrates that the size of the sample,* $n = \Sigma n_i$, *is a random variable. Also, in this case,* $\frac{n_i}{N_i}$ *cannot be exactly constant. One should consider whether there is an appreciable bias in the estimator (4.9). Use (4.6) instead of (4.9) unless there is assurance that any bias in (4.9), owing to unequal probabilities of the ssu's being in the sample, is negligible.*

4.4 SELECTION OF PSU'S WITH PPS

Consider a sample of m psu's selected with replacement and with selection probabilities $P_1, P_2 \ldots, P_n$ (See section 1.1.2 in Chapter I). Let n_i be the size of a simple random sample of ssu's that is to be selected from the i^{th} psu in the event that it is selected. If, by chance, the i^{th} psu is selected a second time another sample of n_i ssu's is selected. For a sample of n psu's the estimator is:

$$\hat{y} = \left(\frac{1}{N}\right)\left(\frac{1}{m}\right) \sum_{i}^{m} \frac{N_i \bar{y}_i}{p_i} \qquad (4.12)$$

Remember to interpret "i" as an index of the psu's selected by the m random draws. Notice that $N_i\bar{\bar{y}}_i$ is an estimate of a psu total and that $\frac{N_i\bar{\bar{y}}_i}{p_i}$ is an estimate of the population total, Y, based on a sample of one psu and a simple random sample of n_i ssu's within it. Thus, there are m estimates of Y and $(\frac{1}{m})\sum_i^m \frac{N_i\bar{\bar{y}}_i}{p_i}$ is an average of these estimates. The factor $\frac{1}{N}$ makes $\hat{y}$ an estimator of $\bar{\bar{Y}}$. The variance of $\hat{y}$, in Eq. 4.12, is:

$$V(\hat{y}) = \frac{1}{m}\left[\sigma_1^2 + \frac{1}{N^2}\sum_i^M (\frac{1}{P_i})(1-f_{2i})\frac{N_i^2 S_{2i}^2}{n_i}\right] \qquad (4.13)$$

where $$\sigma_1^2 = \frac{1}{N^2}\sum_i^M P_i(\frac{Y_i}{P_i} - Y)^2$$

and $$S_{2i}^2 = \frac{\sum_j^{N_i}(Y_{ij} - \bar{\bar{Y}}_i)^2}{N_i - 1}$$

Exercise 4.15 Compare $\frac{\sigma_1^2}{m}$ *in 4.13 with the variance of* $\hat{y}_4$ *in Table 1.1, using the alternative expression for* σ_4^2 *in the variance of* $\hat{y}_4$. *Change the notation used in Chapter 1 to conform to the notation used for psu's. This gives:*

$$V(\hat{y}_4) = (\frac{1}{m})(\frac{1}{M^2})\sum_i^m P_i(\frac{Y_i}{P_i} - Y)^2$$

Why is this expression for $V(\hat{y}_4)$ *different from the between psu part of the variance in 4.13? In terms of the notation for two-stage sampling* $\hat{y}_4$ *is an estimate of* Y *rather than* $\bar{\bar{Y}}$. *Change* $\hat{y}_4$ *so it will be an estimator of* $\bar{\bar{Y}}$ *and make the corresponding change in* $V(\hat{y}_4)$. *Your answer should agree exactly with* $\frac{\sigma_1^2}{m}$ *in (4.13).*

Notice the correspondence between S_1^2 in Eq. 4.7 and the variance of $\hat{y}_1$, plan 1, in Chapter I; also, notice the correspondence between σ_i^2 in Eq. 4.13 and the variance of $\hat{y}_4$, plan 4, in Chapter I. The discussion in Chapter I of the efficiency of plan 4 compared to plan 1 is relevant to the first stage of sampling. If N_i is a good measure of size, σ_1^2 will be considerably less than S_1^2.

Compare the components of variance in Eq. 4.7 and Eq. 4.13 which pertain to the second stage of sampling. The only difference is a reflection of the difference in the probabilities of selection at the first stage. When the probabilities are equal, $P_i = \frac{1}{M}$ and substituting $\frac{1}{M}$ for P_i in 4.13 gives 4.7.

In Eq. 4.4, f_{ij} was expressed as the probability that any given ssu has of being in a sample assuming the sample at both stages was simple random sampling without replacement. This equation now needs modification to be in accord with sampling at the first stage with unequal probability and with replacement. An appropriate probability equation is:

$$f'_{ij} = P_i f_{2i} = P_i \frac{n_i}{N_i} \tag{4.14}$$

where P_i is the selection probability, at any given random draw, for the i^{th} psu in the population,

f_{2i} as defined before, is the sampling fraction within the i^{th} psu of the population, and

f_{ij} is the probability which the j^{th} ssu in the i^{th} psu of the population has of being in a sample obtained by selecting <u>one</u> psu with pps and selecting a simple random sample of n_i ssu's within the selected psu.

It is in the context of the probability Eq. 4.14 that the estimator, 4.12 and its variance, 4.13, are applicable, assuming m independent random selections of psu's.

The estimator, Eq. 4.12, and its variance, Eq. 4.13, are for any given set of selection probabilities at the first stage and any given set of sample sizes, n_i, at the second stage. An important special case exists when f'_{ij}, in Eq. 4.14 is held constant and when the psu's are selected with probabilities proportional to N_i, that is, when $P_i = \frac{N_i}{N}$. By letting f' be the constant value of f'_{ij}, we obtain the following results from Eq. 4.14:

$$n_i = f'N = \bar{n}$$

and

$$f_{2i} = \frac{\bar{n}}{N_i}$$

That is, the sample size within a psu is constant, and, since f'_{ij} is also constant, the sample is self-weighted.

The estimator and its variance become:

$$\hat{\bar{y}} = \frac{\Sigma\Sigma y_{ij}}{n} \tag{4.15}$$

and $$V(\hat{\bar{y}}) = \frac{1}{m}\left[\sigma_1^2 + \frac{1}{\bar{n}}\frac{1}{N}\sum_i^M N_i(1 - f_{2i})S_{2i}^2\right] \tag{4.16}$$

where $$\sigma_1^2 = \frac{1}{N}\sum_i^M N_i(\bar{\bar{Y}}_i - \bar{\bar{Y}})^2$$

For computational purposes one might use:

$$\sigma_1^2 = \frac{1}{N}\sum_i^M \frac{Y_i^2}{N_i} - (\frac{Y}{N})^2$$

and $$\frac{1}{N}\sum_i^M N_i(1 - f_{2i})S_{2i}^2 = \frac{1}{N}\Sigma N_i S_{2i}^2 - \frac{\bar{n}}{N}\Sigma S_{2i}^2$$

Exercise 4.16 Show that Eqs. 4.12 and 4.13 reduce to 4.15 and 4.16 when $P_i = \frac{N_i}{N}$ *and* $n_i = \bar{n}$.

When the N_i are not known, estimates of N_i or a suitable measure of size might be used in place of N_i. In this case, assuming $f'_{ij} = f'$, the sampler would choose a value of f' such that f'N is the desired average size of sample from a psu. Since the selection probabilities for psu's are known, the second-stage sampling fraction $f_{2i} = \frac{f'}{P_i}$ would be calculated for each selected psu. Application of these second-stage sampling fractions gives a self-weighted sample. The n_i will be nearly equal if the measure of size is close to being proportional to N_i. The estimator, Eq. 4.12, and its variance, Eq. 4.13, are applicable. They could be modified by making use of the fact that $p_i f_{2i} = p_i \frac{n_i}{N_i} = f'$.

4.4.1 NUMERICAL EXAMPLE

Exercise 4.17 With reference to the apple tree example, we found for simple random sampling at both stages that the sampling variance was 797.6 *when* m=2 *and* $n_i=\bar{n}=2$ *(See Table 4.4). For comparative purposes, find the sampling variance for* m=2 *and* $\bar{n}=2$ *when the trees are selected with probabilities proportional to* N_i. *The data needed are found in Table 4.3, columns headed* N_i, Y_i, *and* S_{2i}^2. *Find the values of* σ_1^2, $\frac{1}{N}\Sigma N_i S_{2i}^2$, *and* $\frac{1}{N}\Sigma S_{2i}^2$, *then compute the variance of* $\hat{y}$ *for* m=2 *and* $\bar{n}=2$. *Ans.* 532.6.

Substituting results from exercise 4.16 in Eq. 4.16 gives:

$$V(\hat{y}) = \frac{439.7}{m} + \frac{1367 - \bar{n}\ (57.99}{m\bar{n}} \qquad (4.17)$$

For m=2 the between psu variance, $\frac{439.74}{2}$ = 219.9, compares with 458.6 (see Table 4.4) when two psu's are selected with equal probability. As indicated by this result, selecting psu's with pps is often very important in reducing the between psu component of variance. For m=2 and $\bar{n}=2$ the within psu component in Eq. 4.17 is equal to 312.8 which compares with two other results that were obtained when the trees are selected with equal probabilities: 339.0 when $n_i = 2$, and 311.4 when $\frac{n_i}{N_i}$ is constant and $\bar{n}=2$. The first result, 339.0, was recorded in Table 4.4 and the second, 311.4, is readily obtained by Eq. 4.11.

Suppose that one tree is selected with probability $\frac{N_i}{N}$ and that one branch is selected from it with equal probability. In this case, m=1, and $\bar{n}$=1, and the variance of $\hat{y}$ according to variance formula 4.17 is 1748.8. The probability of selecting any given branch in the population is $(\frac{N_i}{N})(\frac{1}{N_i})$. This is a special case of two-stage sampling that is the same as a single-stage, simple random sample of one branch. We found earlier that the variance among the 135 branches was 1762. The exact variance for a simple random sample of one branch is:

$$(\frac{135 - 1}{135}) \quad \frac{1762}{1} = 1748.8$$

4.5 UNEQUAL PROBABILITY OF SELECTION AT BOTH STAGES

As a further exposition of the theory for two-stage sampling, suppose a sample of trees is selected with replacement and with selection probabilities proportional to trunk size. Also, suppose that the method of sampling at the second stage is the random-path method, RP-PPS, that was discussed in Chapter III. You may recall that the random-path method was presented in the context of sampling with replacement.

When the sampling at both stages is with unequal probability, the estimator of the population total Y is:

$$\hat{y}_t = \frac{\sum_i^m \sum_j^{n_i} \left[\frac{y_{ij}}{p_{ij}}\right]}{n} \tag{4.18}$$

where $n = \sum_i^m n_i$

$$p_{ij} = p_i p(j|i)$$

p_i is the selection probability for the i^{th} psu in the sample, and

$p(j|i)$ is the selection probability for the j^{th} ssu given that its psu has been selected.

Consider the quantity $\frac{y_{ij}}{p_{ij}}$ in the estimator. When the value for a unit in the sample (in this case, y_{ij}) is divided by its selection probability (in this case, p_{ij}) the quotient is an estimate of the population total. Therefore, $\hat{y}_t$ in Eq. 4.18 is an average of n estimates of Y, one estimate from each branch in the sample.

The subscript "t" was added to $\hat{y}$ because it is an estimator of Y rather than $\bar{Y}$. Notice that the estimator does not contain N. In practice, one finds many populations to be sampled where N is unknown. An estimate, $\hat{N}$, of N might be made from a sample and, if needed, $\frac{\hat{y}}{\hat{N}}$ could be used as an estimate of $\bar{\bar{Y}}$. An estimator of N is obtained by substituting "1" for y_{ij} in 4.18.

Exercise 4.18 Suppose, for m=3, *and* n_i=2, *that application of the above method to the apple tree population gives the following sample:*

Population index values of i *and* j		*Sample index values of* i *and* j				
Tree	*Path*	*Tree*	*Path*	p_i	$p(j\|i)$	y_{ij}
1	2-2-3	3	1	0.07035	.15996	59.5
1	4-1	3	2	0.07035	.07779	7.0
3	1-2-1-2	1	1	0.2312	.04864	139.3
3	2-3	1	2	0.2312	.05757	125.0
3	3-1-2	2	1	0.2312	.02081	31.3
3	1-2-1-2	2	2	0.2312	.04864	139.3

Tree No. 3 and path 1-2-1-2 were selected twice. The selection probabilities P_i *were proportional to* X_i*, the trunk sizes which are presented in Table 4.3. Verify the values of* p_i*. For tree No. 3 in the population, the conditional probabilities,* $P(j|i)$*, are the probabilities in Column* P_4 *of Table 3.2. Thus, the above values of* $p(j|i)$ *and* y_{ij} *for the branches in the sample from this tree were taken from Table 3.2. The values of* $p(j|i)$ *and* y_{ij} *for tree No. 1 are from records not reproduced herein. Using Eq. 4.18 as the estimator, calculate the estimate of the total number of apples. The answer is* 7873*, which is an estimate of* 7199*, the total number of apples including "path" apples (See Table 4.3).*

To find the variance of $\hat{y}_t$, refer to Eq. 4.13 and make two modifications:

(1) for the first stage we want $N^2\sigma_1^2$ instead of σ_1^2 because $\hat{y}_t$ is an estimator of Y rather than $\bar{\bar{Y}}$, and

(2) for the second stage, the part of the formula representing the variance of an estimate of Y_i for a simple random sample of n_i needs to be changed. That is, $(1 - f_{2i}) \frac{N_i^2 S_{2i}^2}{n_i}$ needs to be replaced by the corresponding variance for sampling within the i^{th} psu with pps. Also, $\frac{1}{N^2}$ needs to be dropped. This gives:

$$V(\hat{y}_t) = \frac{1}{m}\left[\sum_i^M P_i\left(\frac{Y_i}{P_i} - Y\right)^2 + \sum_i^M\left(\frac{1}{P_i}\right)\frac{S_{ri}^2}{n_i}\right] \qquad (4.19)$$

where $$S_{ri}^2 = \sum_{j}^{N_i} P(j|i)\left(\frac{Y_{ij}}{P(j|i)} - Y_i\right)^2$$

The subscript r signifies random path.

For Tree No. 3 the values of $P(j|i)$ and the values of $\frac{Y_{ij}}{P(j|i)}$ are in columns P_4 and $\hat{Y}_4$ of Table 3.2. From these two columns the value of S_{ri}^2 for tree number 3 can be computed. The answer, 800,000 is recorded along with other values of S_{ri}^2 in the last column of Table 4.3.

Exercise 4.19 When $n_i=\bar{n}$, *the second term in* [] *of Eq. 4.19 becomes* $\frac{1}{\bar{n}} \sum_{i}^{M} \frac{S_{ri}^2}{P_i}$. *In the problem under consideration,* $P_i=\frac{X_i}{X}$ *where* X_i *is trunk size. From the data in Table 4.3, find the value of* $\Sigma\frac{S_{ri}^2}{P_i}$ *and of* $\Sigma P_i\left(\frac{Y_i}{P_i} - Y\right)^2$. *When your results are substituted in 4.19, you should have:*

$$V(\hat{y}_t) = \frac{1}{m}\left[5{,}322{,}000 + \frac{11{,}462{,}000}{\bar{n}}\right]$$

Exercise 4.20 From the sample data given in Exercise 4.16 estimate the total number of terminal branches on the six trees. Ans. 122.0.

When the equation in Exercise 4.19 is divided by N^2 we obtain:

$$V(\hat{y}) = \frac{1}{m}\left[292 + \frac{629}{\bar{n}}\right]$$

To summarize, the following variance equations have been obtained for three alternative two-stage plans for sampling the small population of apple trees:

$$(1) \qquad V(\hat{y}) = \frac{1}{m}\left[917.1 + \frac{(1-f_2)\ 1367}{\bar{n}}\right]$$

for simple random sampling at both stages, where $\frac{n_i}{N_i}$ is constant and equal to f_2 and $1-f_1$ was assumed to be equal to 1,

$$(2) \qquad V(\hat{y}) = \frac{1}{m}\left[439.7 + \frac{1367 - \bar{n}(58.0)}{\bar{n}}\right]$$

for sampling trees with probability proportional to N_i and a simple random sample of $\bar{n}$ branches from each selected tree, and

$$(3) \qquad V(\hat{y}) = \frac{1}{m}\left[292 + \frac{629}{\bar{n}}\right]$$

for sampling trees with probability proportional to X_i (trunk size) and application of the RP-PPS method to the sample trees.

The results are too limited to provide a basis for generalization.

Table 4.1 Representation of Population Data for Two Stage Sampling[1/]

psu	ssu: 1 ... j ... N_i	psu total	psu mean	Within psu variances
1	$Y_{11} \cdots Y_{1j} \cdots Y_{1N_1}$	Y_1	$\bar{\bar{Y}}_1$	$S_{21}^2 = \frac{\sum_j (Y_{1j} - \bar{\bar{Y}}_1)^2}{N_1 - 1}$
.				
.				
i	$Y_{i1} \cdots Y_{ij} \cdots Y_{iN_i}$	Y_i	$\bar{\bar{Y}}_i$	$S_{2i}^2 = \frac{\sum_j (Y_{ij} - \bar{\bar{Y}}_i)^2}{N_i - 1}$
.				
.				
M	$Y_{M1} \cdots Y_{Mj} \cdots Y_{MN_M}$	Y_M	$\bar{\bar{Y}}_M$	$S_{2M}^2 = \frac{\sum_j (Y_{Mj} - \bar{\bar{Y}}_M)^2}{N_M - 1}$

[1/] A single bar "-" is used for an average of psu totals and a double bar "=" indicates an average of secondary units. A subscript 1 or 2 affixed to S^2 indicates first or second stage variance. See definitions below.

Y_{ij} is the value of the characteristic Y for the j^{th} ssu in the i^{th} psu,

$Y_i = \sum_j^{N_i} Y_{ij}$ is the total of Y for the i^{th} psu,

$Y = \sum_i^M \sum_j^{N_i} Y_{ij} = \sum_i^M Y_i$ is the total of Y for the population,

M is the number of psu's in the population,

N_i is the population number of ssu's in the i^{th} psu,

$N = \sum_i^M N_i$ is the number of ssu's in the population,

$\bar{Y} = \frac{Y}{M}$ is the population mean per psu,

$\bar{\bar{Y}} = \frac{Y}{N}$ is the population mean per ssu,

$\bar{\bar{Y}}_i = \frac{Y_i}{N_i}$ is the average value of Y per ssu in the i^{th} psu,

$\bar{N} = \frac{N}{M}$ is the average number of ssu's per psu,

$S_{2i}^2 = \sum_j^{N_i} \frac{(Y_{ij} - \bar{\bar{Y}}_i)^2}{N_i - 1}$ is the variance among ssu's in the i^{th} psu, and

$S_1^2 = (\frac{1}{\bar{N}^2}) \frac{\sum_i^M (Y_i - \bar{Y})^2}{M-1}$ is the variance among psu's on the basis of one ssu.

Table 4.2 Components of Variation for a Hypothetical Population

psu	1	2	3	4	5	Y_i	$\bar{\bar{Y}}_i$	S_{2i}^2
	Values of Y_{ij}							
1	67	45	51	20	35	218	43.6	308.8
2	32	27	82	39	18	198	39.6	620.3
3	14	25	21	30	28	118	23.6	40.3
4	55	48	72	63	88	326	65.2	242.7
						$Y = 860$	$\bar{\bar{y}} = 43.0$	
	Values of $(Y_{ij} - \bar{\bar{Y}})$							
1	24	2	8	-23	-8			
2	-11	-16	39	-4	-25			
3	-29	-18	-22	-13	-15			
4	12	5	29	20	45			
	Values of $(\bar{\bar{Y}}_i - \bar{\bar{Y}})$							
1	0.6	0.6	0.6	0.6	0.6			
2	-3.4	-3.4	-3.4	-3.4	-3.4			
3	-19.4	-19.4	-19.4	-19.4	-19.4			
4	22.2	22.2	22.2	22.2	22.2			
	Values of $(Y_{ij} - \bar{\bar{Y}}_i)$							
1	23.4	1.4	7.4	-23.6	-8.6			
2	-7.6	-12.6	42.4	-0.6	-21.6			
3	-9.6	1.4	-2.6	6.4	4.4			
4	-10.2	-17.2	6.8	-2.2	22.8			

$S^2 = 487.053$ is the variance among the 20 values of $(Y_{ij} - \bar{\bar{Y}})$

$S_1^2 = 293.707$ is the variance among the 4 values of $(\bar{\bar{Y}}_i - \bar{\bar{Y}})$

$S_2^2 = 303.025$ is the average of the variances of $(Y_{ij} - \bar{\bar{Y}}_i)$ within psu's. Within the first psu the variance is:

$$\frac{23.4^2 + 1.4^2 + 7.4^2 + (-23.6)^2 + (-8.6)^2}{4} = 308.8.$$

Table 4.3 Summary Data for Six Apple Trees [1/]

Tree	No. of Terminal Branches N_i	No. of Apples on Terminal Branches Y_i	Within Tree Variance DS-EP S^2_{2i}	Trunk Size in Sq. In. X_i	Total No. of Apples on Tree Y'_i	Within Tree Variance RP-PPS S^2_{ri}
1	13	213	259	7.0	214	22,000
2	27	1,388	1,147	20.0	1,448	478,000
3	26	1,850	2,184	23.0	1,901	800,000
4	20	1,592	3,106	16.5	1,658	350,000
5	19	402	241	13.5	403	79,000
6	30	1,528	892	19.5	1,575	513,000
Total	135	6,973		99.5	7,199	

[1/] The values of N_i, Y_i, and S^2_{2i} are from Table 2.6. The values of Y_i and S^2_{2i} are labeled Y_h and S^2_{Yh} in Table 2.6. "Path apples" are not included in Y_i and S^2_{2i}. The values of Y'_i and S^2_{ri} include the path apples and are taken from Table 3.3. The subscript "r" refers to random path.

DS-EP and RP-PPS refer to the method of sampling a tree as discussed in Chapter III.

Table 4.4 Variances for Alternative Sample Allocations

	m	$\bar{n}$	$V(\hat{y})$	Components: Among psu's [1/]	Components: Within psu's
(1)	1	4	1225.8	917.1	308.7
(2)	2	2	797.6	458.6	339.0
(3)	4	1	583.5	229.3	354.2

[1/] Assumes f_1 is negligible.

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